Best NET-CC: Extension of BCC search to GW interferometer networks

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Talk Outline

• Detection problem of arbitrary chirp with known phase: Max LR methods
  \[ \Rightarrow \] Linear Least SQuare Problem can be ill-posed!
  Consequence for binary inspiral detection?

• Formulate network max LLR with Synthetic Streams
  \[ \Rightarrow \] Implementation of Best Net-CC
Talk Outline

- Detection problem of arbitrary chirp with known phase: Max LR methods
  $\Rightarrow$ Linear Least Squares Problem can be ill-posed!
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- Formulate network max LLR with Synthetic Streams
  $\Rightarrow$ Implementation of Best Net-CC

Best CC (BCC) – Time frequency based detection of unmodelled chirps

Look for TF track MLR
$=\text{Longest Path in TF}$
Signal Model: Smooth chirp

GW polarisations:

\[ h_+ = A \frac{(1 + \cos^2 \epsilon)}{2} \cos(\varphi - \varphi_0) \]

\[ h_x = A \cos \epsilon \sin(\varphi - \varphi_0) \]

- \( A \): amplitude
- \( \varphi_0 \): initial phase
- \( \epsilon \): arbitrary phase
Signal Model: Smooth chirp

GW polarisations:
\[ h_+ = A \frac{(1+\cos^2 \epsilon)}{2} \cos(\varphi - \varphi_0) \]
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A: amplitude, \( \varphi_0 \): initial phase, \( \varphi \): arbitrary phase

Network Signal, \( s \in \mathbb{R}^{Nd} \)

\[ s = \frac{1}{2} \left( \begin{bmatrix} D & D^* \end{bmatrix} \otimes \begin{bmatrix} \Phi^* & \Phi \end{bmatrix} \right) \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \equiv \Pi \mathbb{P} \]
What is SVD of $\Pi = U_\Pi \Sigma_\Pi V_\Pi^H$?

SVD of $\otimes$ is $\otimes$ of SVD

$\Phi$ and $\Phi^*$ are orthogonal
What is SVD of $\Pi = U_\Pi \Sigma_\Pi V^H_\Pi$?

\[
\text{SVD}(\Pi) = \underbrace{\text{SVD}(\Pi)}_{\Sigma_\Pi = \text{diag}(\sigma_1, \sigma_2)} \otimes \underbrace{\text{SVD}(\Phi)}_{I_2}
\]

SVD of $\otimes$ is $\otimes$ of SVD

$\Phi$ and $\Phi^*$ are orthogonal

Condition Number

\[
\text{cond}(\Pi) = \sigma_1 / \sigma_2
\]

$\sigma_1 > \sigma_2$

$\sigma_2$ can be very small $\implies$ $F_+$ and $F_\times$ are collinear. [Klimenko et. al PRD 2005, Rakhmanov CQG 2006]
Detector Networks: $\text{cond } (\Pi)^{-1} = \sigma_2 / \sigma_1 < 0.1$
Maximum of Network LLR

- Maximize Network Likelihood Ratio wrt $\mathbf{P}$:
  \[ \Lambda = -\| \mathbf{x} - \Pi \mathbf{P} \|^2 + \| \mathbf{x} \|^2 \]

Solve Linear LSQ:- Pseudo-inverse of $\Pi$ i.e. $\hat{\mathbf{P}} = V\Pi \Sigma^{-1}_\Pi U^H \mathbf{x}$
Maximum of Network LLR

- Maximize Network Likelihood Ratio wrt \( \Pi \):
  \[
  \Lambda = -\|x - \Pi \|_2^2 + \|x\|_2^2
  \]
  Solve Linear LSQ:- Pseudo-inverse of \( \Pi \) i.e. \( \hat{\Pi} = V\Sigma_{\Pi}^{-1}U_H^T \)

- Network MLR:
  \[
  \Lambda(\hat{\Pi}) = \|U_{\Pi}^H x\|_2^2 = \|(U_D^H \otimes U_{\emptyset}^H)x\|_2^2
  \]

Project data on to \( U_{\emptyset} \) first and then combine with weights

\[ [Pai, Dhurandhar, Bose, PRD 2001] \]
Maximum of Network LLR

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Project data on to $U_D$ and then Matched filtering

\[ X = \begin{bmatrix} x_1 & \ldots & x_d \end{bmatrix}_{N \times d} \]
Synthetic Streams and Null Streams

Network MLR: $\Lambda(\hat{P}) = \|U_{\Pi}^H x\|^2 = \|(U_{\mathbb{D}}^H \otimes U_{\emptyset}^H)x\|^2$

$U_{\mathbb{D}} = \begin{bmatrix} d_1 & d_2 \end{bmatrix}$

$U_{\emptyset} = \begin{bmatrix} \Phi^* & \Phi \end{bmatrix}$

(a) Project data on to $U_{\mathbb{D}}$ = construct synthetic streams

(b) Matched filtering of synthetic streams
Synthetic Streams and Null Streams

Network MLR:
\[ \Lambda(\hat{P}) = \|U_H^T x\|^2 = \|(U_D^H \otimes U_\emptyset^H)x\|^2 \]

\[ U_D = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \quad U_\emptyset = \begin{bmatrix} \Phi^* & \Phi \end{bmatrix} \]

(a) Project data on to \( U_D = \) construct synthetic streams
(b) Matched filtering of synthetic streams

• 2 non-zero singular values \( \Rightarrow \) 2 synthetic streams: \( Y_1 = Xd_1 \) and \( Y_2 = Xd_2 \)

\[ \hat{\Lambda} = \Lambda(\hat{P}) = \frac{|\Phi^H Y_1|^2 + |\Phi^H Y_2|^2}{N} \]
Synthetic Streams and Null Streams

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- \( (d - 2) \) zero singular values \( \Rightarrow \) \( (d - 2) \) orthogonal null streams span null space
  
  Null Stream Def: \( Xd_j = 0, \quad d \geq j > 2 \) (signal only)
Synthetic Streams and Null Streams

Network MLR: \[ \Lambda(\hat{\mathbf{P}}) = \|U_H^H \mathbf{x}\|^2 = \|(U_D^H \otimes U_\emptyset^H) \mathbf{x}\|^2 \]
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2 detectors: No null stream
3 detectors: 1 null stream \( d_3 = d_1 \times d_2 = \{\epsilon_{ijk}d_{1j}d_{2k}\} \)

[Wen, Schutz, CQG, 2005]
Synthetic Streams and Null Streams

Network MLR:  \[
\Lambda(\hat{P}) = \|U_H^T x\|^2 = \|(U_D^H \otimes U_\varnothing^H) x\|^2
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U_D = \begin{bmatrix}
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Null Stream Def: \(Xd_j = 0\), \(d \geq j > 2\) (signal only)

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[Wen, Schutz, CQG, 2005]

In general, for \(d\) detectors, \((d - 2)\) null streams:

\[
d_{1i} = \epsilon_{ijk...n}d_{1j}d_{2k}...d_{(1-1)n}\text{ for } d \geq l > 2
\]
Treating Rank Defficiency

Π might be ill-conditioned \( i.e. \) \( \text{cond}(\Pi) \gg 1 \)

\((\sigma_2 \sim 0 \Rightarrow \sum^{-1} \text{diverges: } Y_2 \text{ insensitive to GW})\)

Ill posed LSQ \( \Rightarrow \) Needs treatment, regularisation
Treating Rank Deficiency

\[ \Pi \text{ might be ill-conditioned } i.e. \ \text{cond}(\Pi) >> 1 \]
\[ (\sigma_2 \sim 0 \Rightarrow \sum^{-1} \text{ diverges : } Y_2 \text{ insensitive to GW}) \]
Ill posed LSQ \Rightarrow \text{Needs treatment, regularisation}

- Truncated SVD approach: \( Y_2 \) adds noise to the statistics, discard it.

\[ \hat{\Lambda}_t = \frac{|\Phi^H Y_1|^2}{N} \]

Truncation criterion \( \text{cond}(\Pi) > \text{SNR} \)
Treating Rank Deficiency

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  \]

  Truncation criterion \( \text{cond}(\Pi) > \text{SNR} \)

- **Tikhonov Regularisation:** similar to [Rakhmanov CQG 2006]
  
  Regularise \( \Lambda = \text{Add a quadratic regulator to } \Lambda \)
  
  \[
  \hat{\Lambda}_r = \frac{1}{N} \left[ |\Phi^H Y_1|^2 + \frac{|\Phi^H Y_2|^2}{\text{cond}(\Pi)} \right]
  \]

  Larger the \( \text{cond}(\Pi) \) smaller is the contribution from \( Y_2 \)
Best Net-CC : Extension of BCC

Signal phase $\Phi$ is unknown : maximising $\hat{\Lambda}$ over possible phases
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Time-frequency (TF) mapping of $\hat{\Lambda}$ –

TF map – Wigner-Ville transform [Chassande-mottin, Pai, IEEE SPL 2005]

+ Simplified template of smooth chirp – 1-dim ridge in WV i.e. $\delta(t, f(t))$

== TF path integral on WV map [Chassande-mottin, Pai, PRD 2006]
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$=\,$ TF path integral on WV map [Chassande-mottin, Pai, PRD 2006]

$$\hat{\Lambda} = \frac{|\Phi^H Y_1|^2 + |\Phi^H Y_2|^2}{N} = \frac{2}{N^2} \sum_n w_Y(t_n, f(t_n))$$

$w_Y = w_{Y_1} + w_{Y_2}$

Maximise $\hat{\Lambda}$ over $\Phi$ = longest path problem in TF map of $w_Y$
Best Net-CC : Extension of BCC

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Diagram:

- Data $X$
- $Y_1$, $Y_2$, DWV
- Cond(π)
- Sky Location + Detector Network
- Longest Path
- TRUNCATED SVD
Best Net-CC : Extension of BCC

Signal phase $\Phi$ is unknown : maximising $\hat{\Lambda}$ over possible phases

Time-frequency (TF) mapping of $\hat{\Lambda}$ –

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+ Simplified template of smooth chirp – 1-dim ridge in WV i.e. $\delta(t, f(t))$

$\hat{\Lambda} = |\Phi^HY_1|^2 + |\Phi^HY_2|^2 = \frac{2}{N^2} \sum_n w_Y(t_n, f(t_n)) \quad w_Y = w_{Y_1} + w_{Y_2}$

Maximise $\hat{\Lambda}$ over $\Phi$ = longest path problem in TF map of $w_Y$

Diagram:

- Data $X$ feeding into $Y_1$ and $Y_2$
- $Y_1$ and $Y_2$ connected to DWV
- DWV output combined with Longest Path
- Longest Path output connected to $T$
- $T = 1/\text{Cond}(\pi)$
- Sky Location + Detector Network
- Tikhonov Regularisation
Simulations: Linear chirp in Gaussian white noise
Concluding Remarks

Chirp detection problem with a detector network in a “new formalism” (LSQ)

- Evidence of degeneracy in the signal model.
  Parameter estimation may be unreliable.
  Need for proper regularisation.
  Possible implication for inspiral search.

- Coherent Network detection == Process 2 synthetic streams
  Straightforward extension of Best CC search == Best Net-CC
  Best Net-CC —- a feasible full sky search of GW chirps
  Fraction of a sec duration $\sim$ few 100 GFlops
Treating Rank Deficiency

\[ \Pi \] might be ill-conditioned \textit{i.e.} cond(\Pi) \gg 1

\( (\sigma_2 \sim 0 \Rightarrow \sum^{-1} \text{ diverges} : Y_2 \text{ insensitive to GW}) \)

Ill posed LSQ \Rightarrow Needs treatment, regularisation

- Truncated SVD approach: \( Y_2 \) adds noise to the statistics, discard it.

\[
\hat{\Lambda}_t = \frac{|\Phi^H Y_1|^2}{N}
\]

parameter estimation??!!

- Tikhonov Regularisation: [Rakhmanov CQG 2006]

  Regularise \( \Lambda \) \( = \) Add a quadratic regulator \( P^H \Omega P \) to \( \Lambda \)

\[
\hat{\Lambda}_r = \frac{1}{N} \left[ |\Phi^H Y_1|^2 + \frac{|\Phi^H Y_2|^2}{\text{cond}(\Pi)} \right]
\]

LSQ estimator \( \Rightarrow \hat{P}_r = (\Pi^H \Pi + \Omega)^{-1} \Pi^H x \)

Larger the \( \text{cond}(\Pi) \) smaller is the contribution from \( Y_2 \)