Best NET-CC: Extension of BCC search to GW interferometer networks

Archana Pai

Albert Einstein Institute, Potsdam

with Éric Chassande-Mottin

AstroParticule et Cosmologie, Paris

CNRS, Observatoire de la Côte de Azur, Nice

(20th Dec. 2006, GWDAW-11, AEI Berlin)

Talk Outline

- Detection problem of arbitrary chirp with known phase : Max LR methods
  $\Longrightarrow$ Linear Least SQuare Problem can be ill-posed!
  Consequence for binary inspiral detection?
- Formulate network max LLR with Synthetic Streams
  $\Longrightarrow$ Implementation of Best Net-CC

Talk Outline

- Detection problem of arbitrary chirp with known phase : Max LR methods

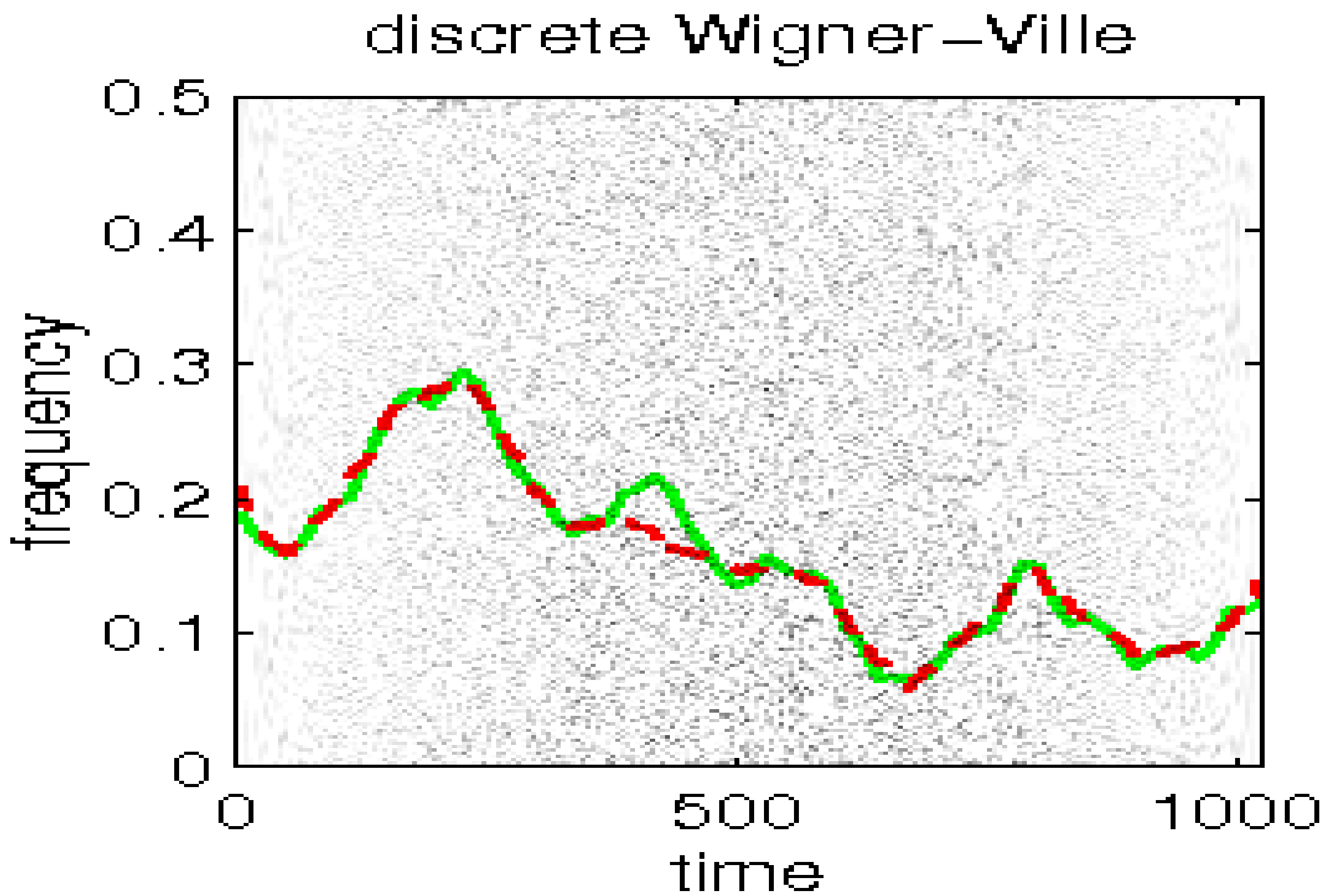
  $\Longrightarrow$ Linear Least SQuare Problem can be ill-posed!

  Consequence for binary inspiral detection?

- Formulate network max LLR with Synthetic Streams

  $\Longrightarrow$ Implementation of Best Net-CC

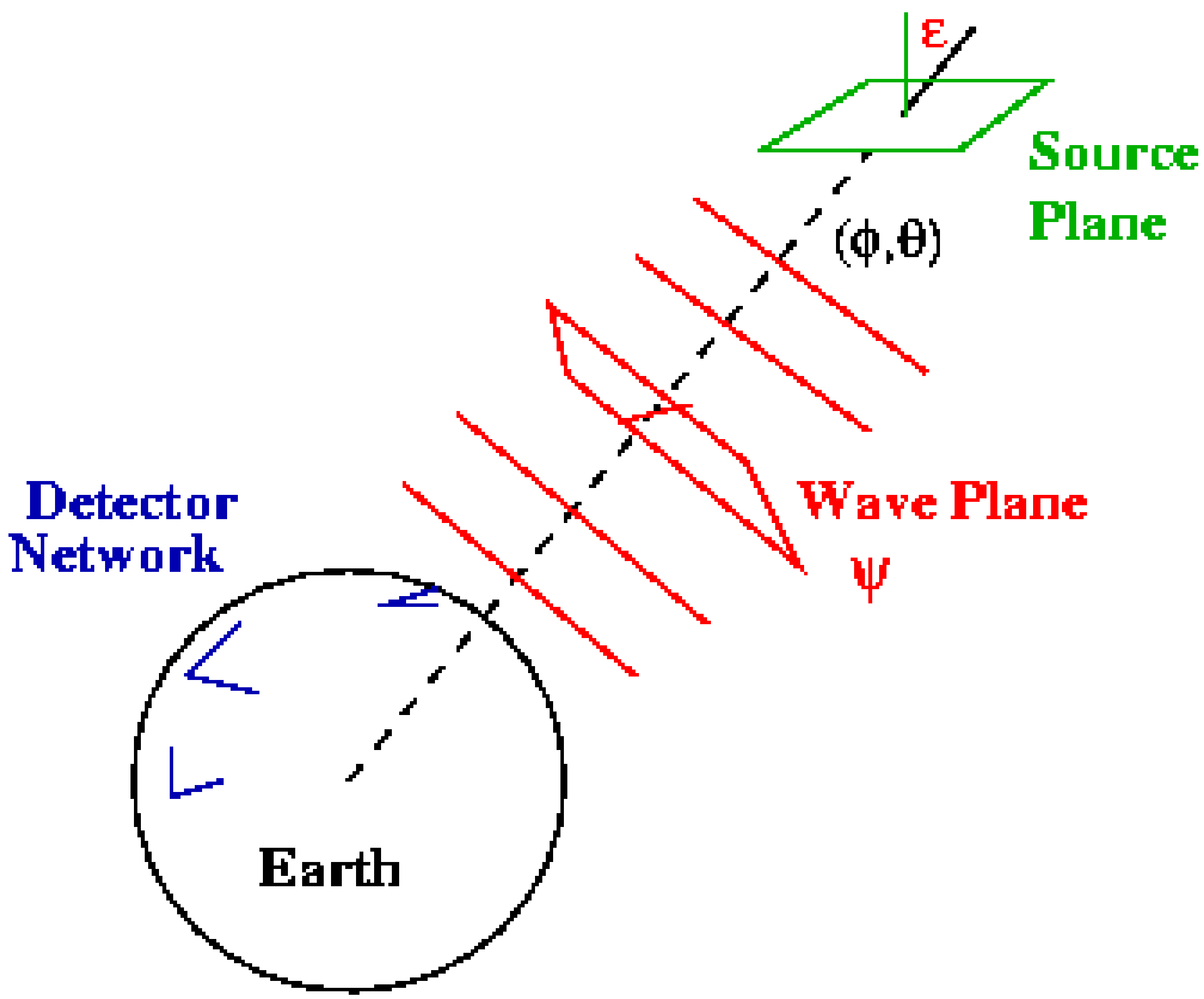
Best CC (BCC) – Time frequency based detection of unmodelled chirps



Look for TF track MLR

= Longest Path in TF

Signal Model : Smooth chirp



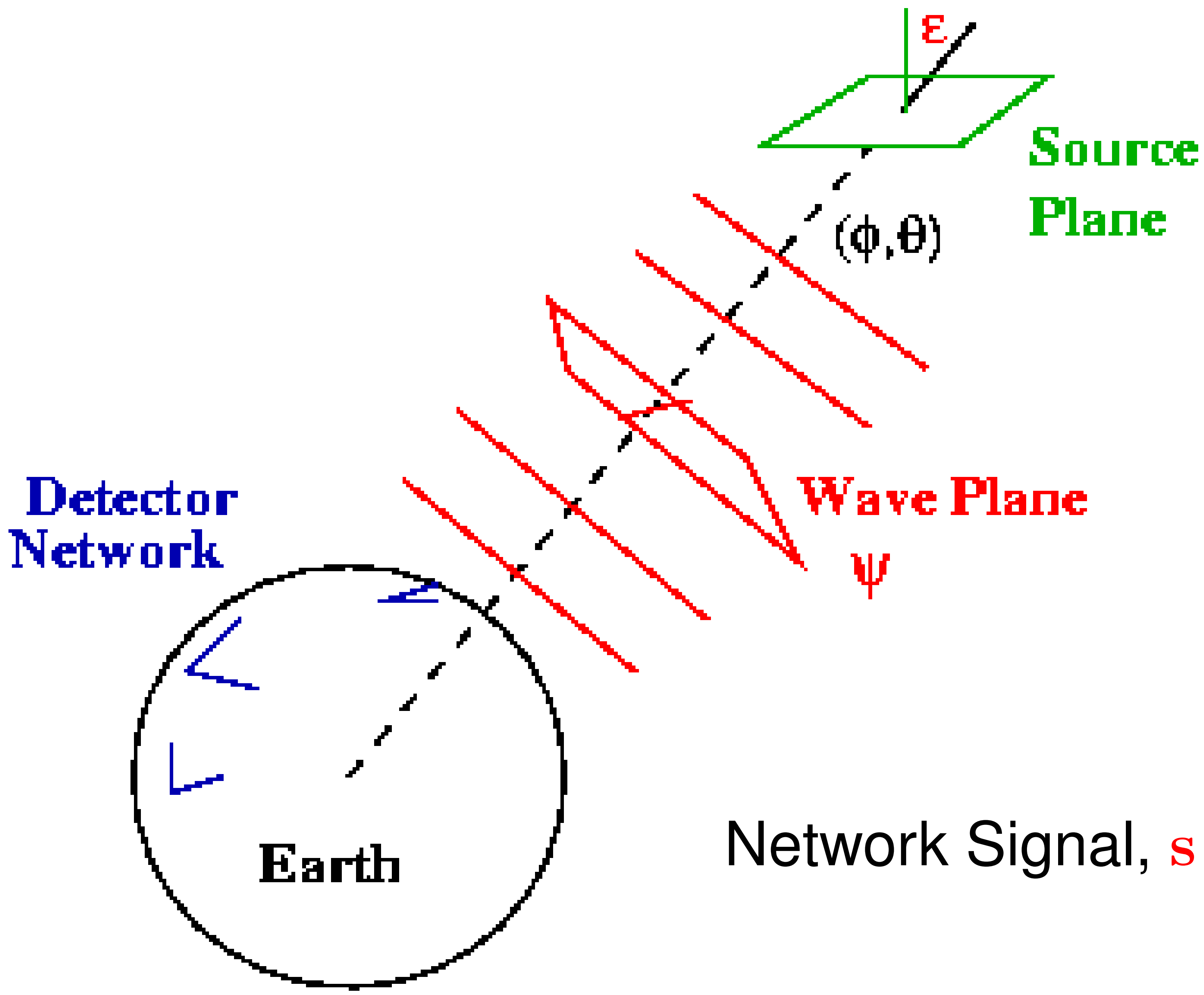
GW polarisations:

$\mathbf{h}_{+} = A\frac{(1+\cos^2\epsilon)}{2}\cos(\varphi - \varphi_0)$

$\mathbf{h}_{\times} = A\cos\epsilon\sin(\varphi - \varphi_0)$

A: amplitude, φ_0: initial phase, φ: arbitrary phase

Signal Model : Smooth chirp



GW polarisations:

$\mathbf{h}_+ = A\frac{(1+\cos^2\epsilon)}{2}\cos(\varphi - \varphi_0)$

$\mathbf{h}_\times = A\cos\epsilon\sin(\varphi - \varphi_0)$

A: amplitude, φ_0: initial phase, φ: arbitrary phase

Network Signal, $\mathbf{s} \in \Re^{Nd}$

$$\mathbf{s} = \frac{1}{2}\left(\underbrace{\begin{bmatrix}\mathbf{D} & \mathbf{D}^*\end{bmatrix}}_{\mathbb{D}} \otimes \underbrace{\begin{bmatrix}\Phi^* & \Phi\end{bmatrix}}_{\varnothing}\right)\underbrace{\begin{pmatrix}p_1\\p_2\\p_3\\p_4\end{pmatrix}}_{\mathbb{P}} \equiv \Pi\,\mathbb{P}$$

Signal Model: Understand Π

What is SVD of $\Pi = U_\Pi \Sigma_\Pi V_\Pi^H$?

SVD of $\otimes$ is $\otimes$ of SVD

$$\text{SVD}(\Pi) = \underbrace{\text{SVD}(\mathbb{D})}_{\Sigma_\Pi = diag(\sigma_1, \sigma_2)} \otimes \underbrace{\text{SVD}(\Phi)}_{\mathbb{I}_\mathbf{2}}$$

Φ and Φ^* are orthogonal

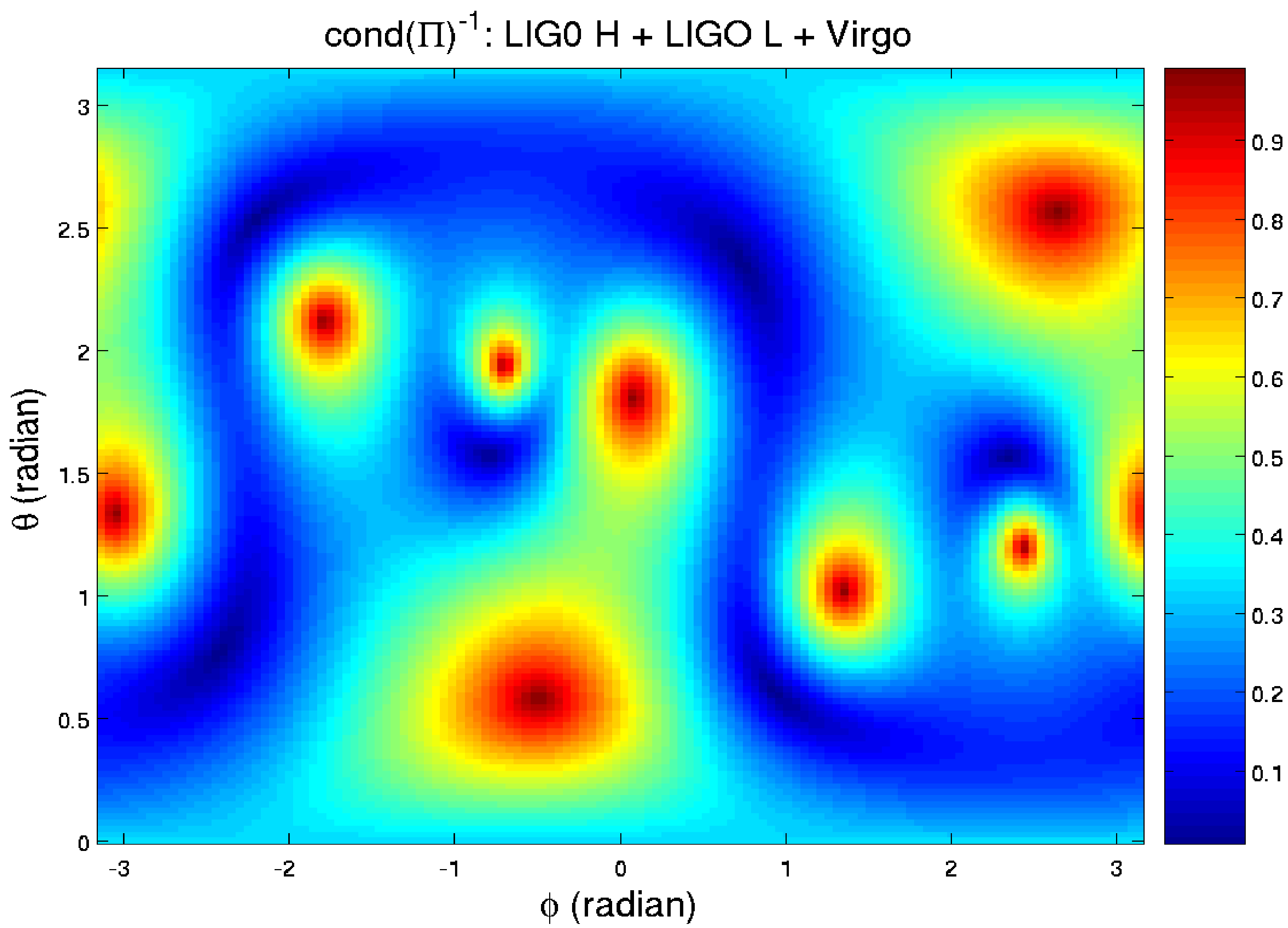
Signal Model: Understand Π

What is SVD of $\Pi = U_\Pi \Sigma_\Pi V_\Pi^H$?

$$\mathrm{SVD}(\Pi) = \underbrace{\mathrm{SVD}(\mathbb{D})}_{\Sigma_\Pi = diag(\sigma_1, \sigma_2)} \otimes \underbrace{\mathrm{SVD}(\Phi)}_{\mathbb{I}_2}$$

SVD of $\otimes$ is $\otimes$ of SVD

Φ and Φ^* are orthogonal



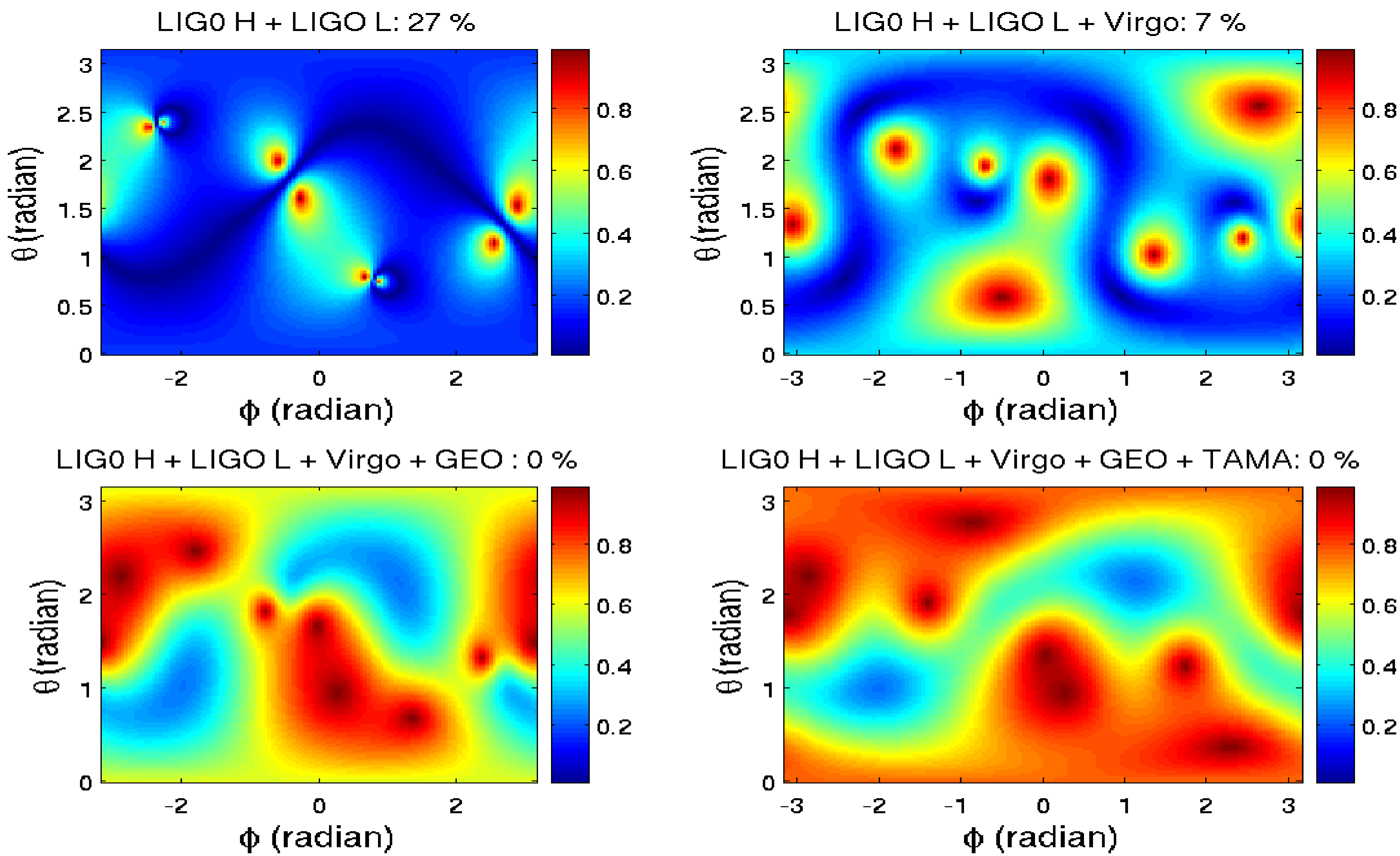
Condition Number

$$\mathrm{cond}(\Pi) = \sigma_1 / \sigma_2$$

$$\sigma_1 > \sigma_2$$

σ_2 can be very small $\Longrightarrow$ $\mathbf{F}_+$ and $\mathbf{F}_\times$ are collinear. [Klimenko et. al PRD 2005, Rakhmanov CQG 2006]

Detector Networks: $\text{cond}\,(\Pi)^{-1} = \sigma_2/\sigma_1 < 0.1$



Maximum of Network LLR

- Maximize Network Likelihood Ratio wrt $\mathbb{P}$:

$$\Lambda = -\|\mathbf{x} - \Pi\mathbb{P}\|^2 + \|\mathbf{x}\|^2$$

Solve Linear LSQ:- Pseudo-inverse of Π *i.e.* $\hat{\mathbb{P}} = V_\Pi \Sigma_\Pi^{-1} U_\Pi^H \mathbf{x}$

Maximum of Network LLR

- Maximize Network Likelihood Ratio wrt $\mathbb{P}$:

$$\Lambda = -\|\mathbf{x} - \Pi\mathbb{P}\|^2 + \|\mathbf{x}\|^2$$

Solve Linear LSQ:- Pseudo-inverse of Π *i.e.* $\hat{\mathbb{P}} = V_\Pi \Sigma_\Pi^{-1} U_\Pi^H \mathbf{x}$

- Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_\Pi^H \mathbf{x}\|^2 = \|(U_\mathbb{D}^H \otimes U_\varnothing^H)\mathbf{x}\|^2$

$$U_\mathbb{D}^H \underbrace{\mathbf{X}^T U_\varnothing} \qquad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_d \end{bmatrix}_{N\times d}$$

Project data on to $U_\varnothing$ first and then combine with weights

[Pai, Dhurandhar, Bose, PRD 2001]

Maximum of Network LLR

- Maximize Network Likelihood Ratio wrt $\mathbb{P}$:

$$\Lambda = -\|\mathbf{x} - \Pi\mathbb{P}\|^2 + \|\mathbf{x}\|^2$$

Solve Linear LSQ:- Pseudo-inverse of Π *i.e.* $\hat{\mathbb{P}} = V_\Pi \Sigma_\Pi^{-1} U_\Pi^H \mathbf{x}$

- Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_\Pi^H \mathbf{x}\|^2 = \|(U_\mathbb{D}^H \otimes U_\varnothing^H)\mathbf{x}\|^2$

$$\underbrace{U_\mathbb{D}^H \mathbf{X}^T}U_\varnothing \qquad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_d \end{bmatrix}_{N\times d}$$

Project data on to $U_\mathbb{D}$ and then Matched filtering

Synthetic Streams and Null Streams

Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_{\Pi}^{H}\mathbf{x}\|^2 = \|(U_{\mathbb{D}}^{H} \otimes U_{\varnothing}^{H})\mathbf{x}\|^2$ $\quad U_{\mathbb{D}} = \left[\begin{array}{cc} \mathbf{d_1} & \mathbf{d_2} \end{array}\right]$

$\underbrace{U_{\mathbb{D}}^{H}\mathbf{X}^{T}}U_{\varnothing}$ $\quad U_{\varnothing} = \left[\begin{array}{cc} \Phi^{*} & \Phi \end{array}\right]$

(a) Project data on to $U_{\mathbb{D}}$ = construct synthetic streams

(b) Matched filtering of synthetic streams

Synthetic Streams and Null Streams

Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_{\Pi}^H \mathbf{x}\|^2 = \|(U_{\mathbb{D}}^H \otimes U_{\varnothing}^H)\mathbf{x}\|^2$ $\quad U_{\mathbb{D}} = \begin{bmatrix} \mathbf{d_1} & \mathbf{d_2} \end{bmatrix}$

$\underbrace{U_{\mathbb{D}}^H \mathbf{X}^T} U_{\varnothing}$ $\quad U_{\varnothing} = \begin{bmatrix} \Phi^* & \Phi \end{bmatrix}$

(a) Project data on to $U_{\mathbb{D}}$ = construct synthetic streams

(b) Matched filtering of synthetic streams

- 2 non-zero singular values $\Rightarrow$ 2 synthetic streams: $\mathbf{Y_1} = \mathbf{X}\mathbf{d_1}$ and $\mathbf{Y_2} = \mathbf{X}\mathbf{d_2}$

$$\hat{\Lambda} = \Lambda(\hat{\mathbb{P}}) = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N}$$

Synthetic Streams and Null Streams

Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_{\Pi}^{H}\mathbf{x}\|^2 = \|(U_{\mathbb{D}}^{H} \otimes U_{\varnothing}^{H})\mathbf{x}\|^2$ $\qquad U_{\mathbb{D}} = \begin{bmatrix} \mathbf{d_1} & \mathbf{d_2} \end{bmatrix}$

$$\underbrace{U_{\mathbb{D}}^{H}\mathbf{X}^{T}}U_{\varnothing} \qquad U_{\varnothing} = \begin{bmatrix} \Phi^* & \Phi \end{bmatrix}$$

(a) Project data on to $U_{\mathbb{D}}$ = construct synthetic streams

(b) Matched filtering of synthetic streams

- 2 non-zero singular values $\Rightarrow$ 2 synthetic streams: $\mathbf{Y_1} = \mathbf{X}\mathbf{d_1}$ and $\mathbf{Y_2} = \mathbf{X}\mathbf{d_2}$

$$\hat{\Lambda} = \Lambda(\hat{\mathbb{P}}) = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N}$$

- $(d-2)$ zero singular values $\Rightarrow$ $(d-2)$ orthogonal null streams span null space

Null Stream Def: $\mathbf{X}\mathbf{d}_j = 0, \quad d \geq j > 2$ (signal only)

Synthetic Streams and Null Streams

Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_{\Pi}^H \mathbf{x}\|^2 = \|(U_{\mathbb{D}}^H \otimes U_{\varnothing}^H)\mathbf{x}\|^2$ $\qquad U_{\mathbb{D}} = \begin{bmatrix} \mathbf{d_1} & \mathbf{d_2} \end{bmatrix}$

$$\underbrace{U_{\mathbb{D}}^H \mathbf{X}^T U_{\varnothing}} \qquad U_{\varnothing} = \begin{bmatrix} \Phi^* & \Phi \end{bmatrix}$$

(a) Project data on to $U_{\mathbb{D}}$ = construct synthetic streams

(b) Matched filtering of synthetic streams

- 2 non-zero singular values $\Rightarrow$ 2 synthetic streams: $\mathbf{Y_1} = \mathbf{X}\mathbf{d_1}$ and $\mathbf{Y_2} = \mathbf{X}\mathbf{d_2}$

$$\hat{\Lambda} = \Lambda(\hat{\mathbb{P}}) = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N}$$

- $(d-2)$ zero singular values $\Rightarrow$ $(d-2)$ orthogonal null streams span null space

Null Stream Def: $\mathbf{X}\mathbf{d}_j = 0, \quad d \geq j > 2$ (signal only)

2 detectors: No null stream $\qquad$ 3 detectors: 1 null stream $\mathbf{d_3} = \mathbf{d_1} \times \mathbf{d_2} = \{\epsilon_{ijk}\mathbf{d_1}_j\mathbf{d_2}_k\}$

[Wen, Schutz, CQG, 2005]

Synthetic Streams and Null Streams

Network MLR: $\Lambda(\hat{\mathbb{P}}) = \|U_{\Pi}^H \mathbf{x}\|^2 = \|(U_{\mathbb{D}}^H \otimes U_{\varnothing}^H)\mathbf{x}\|^2$ $\quad U_{\mathbb{D}} = \begin{bmatrix} \mathbf{d_1} & \mathbf{d_2} \end{bmatrix}$

$\underbrace{U_{\mathbb{D}}^H \mathbf{X}^T U_{\varnothing}}$ $\quad U_{\varnothing} = \begin{bmatrix} \Phi^* & \Phi \end{bmatrix}$

(a) Project data on to $U_{\mathbb{D}}$ = construct synthetic streams

(b) Matched filtering of synthetic streams

- 2 non-zero singular values $\Rightarrow$ 2 synthetic streams: $\mathbf{Y_1} = \mathbf{X d_1}$ and $\mathbf{Y_2} = \mathbf{X d_2}$

$$\hat{\Lambda} = \Lambda(\hat{\mathbb{P}}) = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N}$$

- $(d-2)$ zero singular values $\Rightarrow$ $(d-2)$ orthogonal null streams span null space

Null Stream Def: $\mathbf{X d}_j = 0, \quad d \geq j > 2$ (signal only)

2 detectors: No null stream

3 detectors: 1 null stream $\mathbf{d_3} = \mathbf{d_1} \times \mathbf{d_2} = \{\epsilon_{ijk} \mathbf{d_1}_j \mathbf{d_2}_k\}$

[Wen, Schutz, CQG, 2005]

In general, for d detectors, $(d-2)$ null streams:

$$\mathbf{d_l}_i = \epsilon_{ijk\ldots n} \mathbf{d_1}_j \mathbf{d_2}_k \ldots \mathbf{d_{(l-1)}}_n \quad \text{for } d \geq l > 2$$

Treating Rank Defficiency

Π might be ill-conditioned *i.e.* $\mathrm{cond}(\Pi) >> 1$

($\sigma_2 \sim 0 \Rightarrow \sum^{-1}$ diverges : $\mathbf{Y_2}$ insensitive to GW)

Ill posed LSQ $\Rightarrow$ Needs treatment, regularisation

Treating Rank Defficiency

Π might be ill-conditioned *i.e.* $\mathrm{cond}(\Pi) >> 1$
($\sigma_2 \sim 0 \Rightarrow \sum^{-1}$ diverges : $\mathbf{Y_2}$ insensitive to GW)
Ill posed LSQ $\Rightarrow$ Needs treatment, regularisation

- Truncated SVD approach: $\mathbf{Y_2}$ adds noise to the statistics, discard it.

$$\hat{\Lambda}_t = \frac{|\Phi^H \mathbf{Y_1}|^2}{N}$$

Truncation criterion $\mathrm{cond}(\Pi) > \mathrm{SNR}$

Treating Rank Defficiency

Π might be ill-conditioned *i.e.* $\mathrm{cond}(\Pi) >> 1$
($\sigma_2 \sim 0 \Rightarrow \sum^{-1}$ diverges : $\mathbf{Y_2}$ insensitive to GW)
Ill posed LSQ $\Rightarrow$ Needs treatment, regularisation

- Truncated SVD approach: $\mathbf{Y_2}$ adds noise to the statistics, discard it.

$$\hat{\Lambda}_t = \frac{|\Phi^H \mathbf{Y_1}|^2}{N}$$

  Truncation criterion $\mathrm{cond}(\Pi) > \mathrm{SNR}$

- Tikhonov Regularisation: similar to [Rakhmanov CQG 2006]
  Regularise Λ == Add a quadratic regulator to Λ

$$\hat{\Lambda}_r = \frac{1}{N}\left[|\Phi^H \mathbf{Y_1}|^2 + \frac{|\Phi^H \mathbf{Y_2}|^2}{\mathrm{cond}(\Pi)}\right]$$

  Larger the $\mathrm{cond}(\Pi)$ smaller is the contribution from $\mathbf{Y_2}$

Best Net-CC : Extension of BCC

Signal phase Φ is unknown : maximising $\hat{\Lambda}$ over possible phases

Best Net-CC : Extension of BCC

Signal phase Φ is unknown : maximising $\hat{\Lambda}$ over possible phases

Time-frequency (TF) mapping of $\hat{\Lambda}$ –

TF map – Wigner-Ville transform [Chassande-mottin, Pai, IEEE SPL 2005]

$+$ Simplified template of smooth chirp – 1-dim ridge in WV *i.e.* $\delta(t, f(t))$

$==$ TF path integral on WV map [Chassande-mottin, Pai, PRD 2006]

Best Net-CC : Extension of BCC

Signal phase Φ is unknown : maximising $\hat{\Lambda}$ over possible phases

Time-frequency (TF) mapping of $\hat{\Lambda}$ –

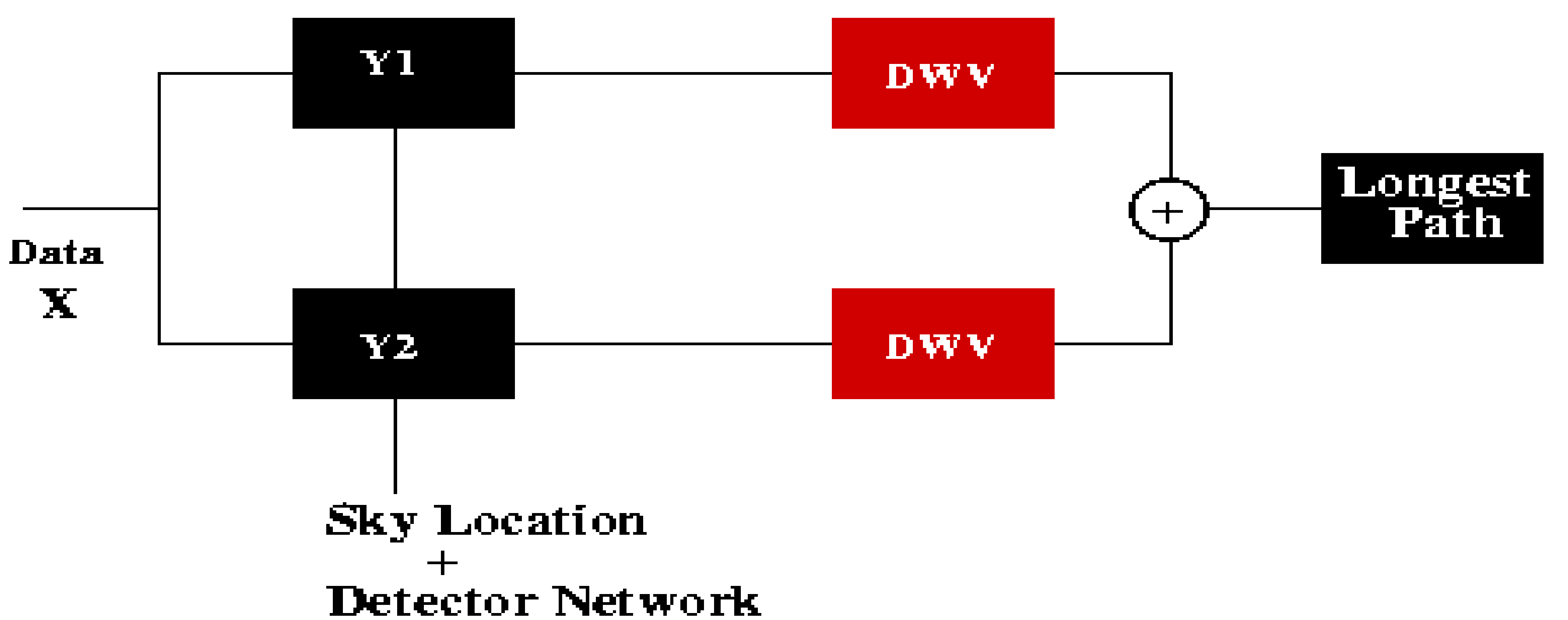
TF map – Wigner-Ville transform [Chassande-mottin, Pai, IEEE SPL 2005]

$+$ Simplified template of smooth chirp – 1-dim ridge in WV *i.e.* $\delta(t, f(t))$

$==$ TF path integral on WV map [Chassande-mottin, Pai, PRD 2006]

$$\hat{\Lambda} = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N} = \frac{2}{N^2} \sum_n w_{\mathbf{Y}}(t_n, f(t_n)) \qquad w_{\mathbf{Y}} = w_{\mathbf{Y_1}} + w_{\mathbf{Y_2}}$$

Maximise $\hat{\Lambda}$ over Φ = longest path problem in TF map of $w_{\mathbf{Y}}$



Best Net-CC : Extension of BCC

Signal phase Φ is unknown : maximising $\hat{\Lambda}$ over possible phases

Time-frequency (TF) mapping of $\hat{\Lambda}$ –

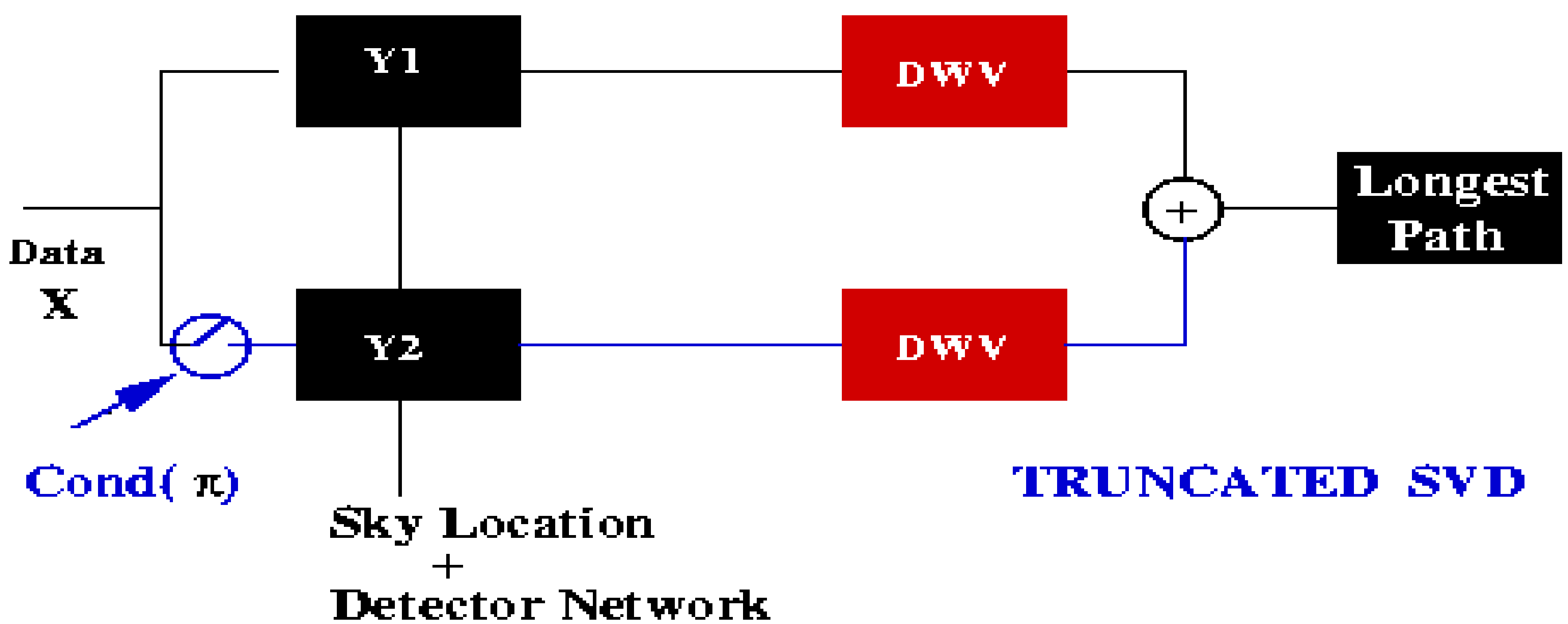
TF map – Wigner-Ville transform [Chassande-mottin, Pai, IEEE SPL 2003]

$+$ Simplified template of smooth chirp – 1-dim ridge in WV *i.e.* $\delta(t, f(t))$

$==$ TF path integral on WV map [Chassande-mottin, Pai, PRD 2006]

$$\hat{\Lambda} = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N} = \frac{2}{N^2} \sum_n w_{\mathbf{Y}}(t_n, f(t_n)) \qquad w_{\mathbf{Y}} = w_{\mathbf{Y_1}} + w_{\mathbf{Y_2}}$$

Maximise $\hat{\Lambda}$ over Φ = longest path problem in TF map of $w_{\mathbf{Y}}$



Best Net-CC : Extension of BCC

Signal phase Φ is unknown : maximising $\hat{\Lambda}$ over possible phases

Time-frequency (TF) mapping of $\hat{\Lambda}$ –

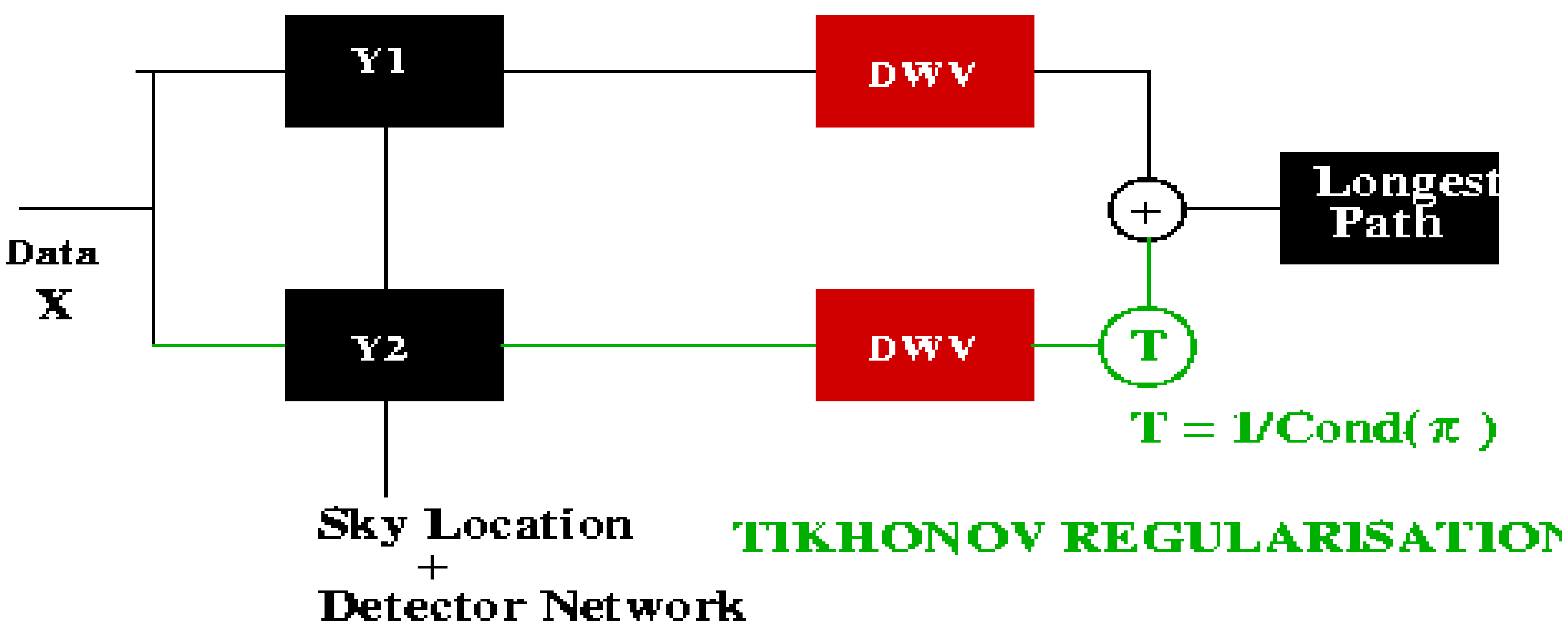
TF map – Wigner-Ville transform [Chassande-mottin, Pai, IEEE SPL 2003]

$+$ Simplified template of smooth chirp – 1-dim ridge in WV *i.e.* $\delta(t, f(t))$

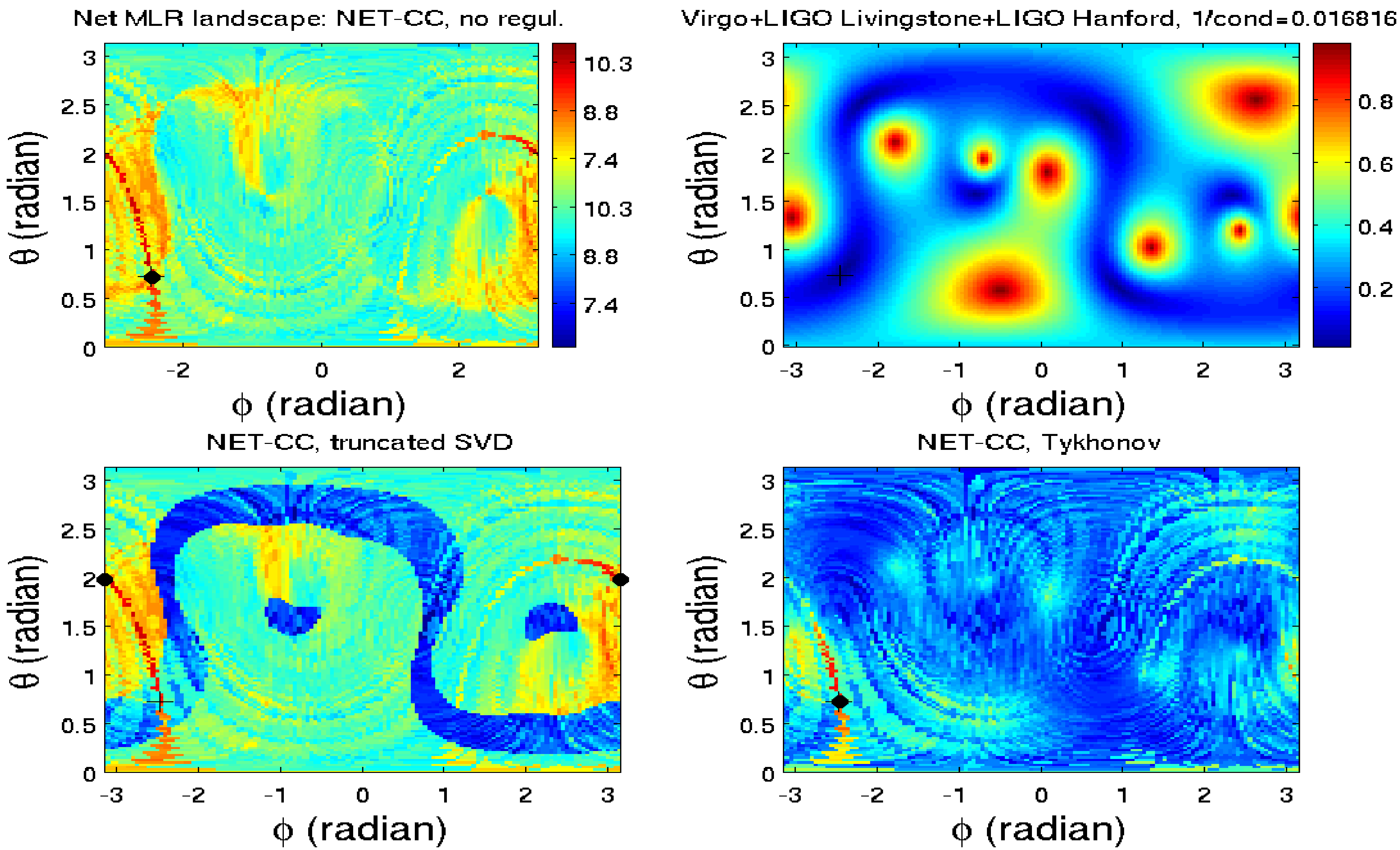
$==$ TF path integral on WV map [Chassande-mottin, Pai, PRD 2006]

$$\hat{\Lambda} = \frac{|\Phi^H \mathbf{Y_1}|^2 + |\Phi^H \mathbf{Y_2}|^2}{N} = \frac{2}{N^2} \sum_n w_{\mathbf{Y}}(t_n, f(t_n)) \qquad w_{\mathbf{Y}} = w_{\mathbf{Y_1}} + w_{\mathbf{Y_2}}$$

Maximise $\hat{\Lambda}$ over Φ = longest path problem in TF map of $w_{\mathbf{Y}}$



Simulations: Linear chirp in Gaussian white noise



Concluding Remarks

Chirp detection problem with a detector network in a “new formalism” (LSQ)

- Evidence of degeneracy in the signal model.
  Parameter estimation may be unreliable.
  Need for proper regularisation.
  Possible implication for inspiral search.

- Coherent Network detection == Process 2 synthetic streams
  Straightforward extension of Best CC search == Best Net-CC
  Best Net-CC —- a feasible full sky search of GW chirps
  Fraction of a sec duration $\sim$ few 100 GFlops

Treating Rank Defficiency

Π might be ill-conditioned *i.e.* $\mathrm{cond}(\Pi) >> 1$

($\sigma_2 \sim 0 \Rightarrow \sum^{-1}$ diverges : $\mathbf{Y_2}$ insensitive to GW)

Ill posed LSQ $\Rightarrow$ Needs treatment, regularisation

- Truncated SVD approach: $\mathbf{Y_2}$ adds noise to the statistics, discard it.

$$\hat{\Lambda}_t = \frac{|\Phi^H \mathbf{Y_1}|^2}{N}$$

parameter estimation??!!

- Tikhonov Regularisation: [Rakhmanov CQG 2006]

Regularise Λ == Add a quadratic regulator $\mathbb{P}^H \Omega \mathbb{P}$ to Λ

$$\hat{\Lambda}_r = \frac{1}{N}\left[|\Phi^H \mathbf{Y_1}|^2 + \frac{|\Phi^H \mathbf{Y_2}|^2}{\mathrm{cond}(\Pi)}\right]$$

LSQ estimator $\Rightarrow \hat{\mathbb{P}}_r = (\Pi^H \Pi + \Omega)^{-1} \Pi^H \mathbf{x}$

Larger the $\mathrm{cond}(\Pi)$ smaller is the contribution from $\mathbf{Y_2}$